AIAA 82-4181

Chordwise and Compressibility Corrections for Arbitrary Planform Slender Wings

Daniel Levin*
NASA Ames Research Center, Moffett Field, Calif.
and
Arnan Seginer†
Technion—Israel Institute of Technology, Haifa, Israel

The Lomax and Sluder method for adapting slender-wing theory to delta or rectangular wings by making chordwise and compressibility corrections is extended to cover wings of any arbitrary planform in subsonic and supersonic flows. The numerical accuracy of the present work is better than that of the Lomax-Sluder results. Comparison of the results of this work with those of the vortex-lattice method and Kernel function method for a family of Gothic and arrowhead wings shows good agreement. A universal curve is proposed for the evaluation of the lift coefficient of a low aspect ratio wing of an arbitrary planform in subsonic flow. The location of the center of pressure can also be estimated.

Nomenclature

AR	= aspect ratio
a_0,a_1,a_2	= coefficients in Eq. (12)
C_{L_α}	= lift curve slope
c	= chord
$f(X)$	= longitudinal correction function
$I(X,X')$	= spanwise integrals [Eqs. (8) and (11)]
M	= Mach number
m	= slope of the leading edge relative to the centerline
$\Delta p/q$	= pressure coefficient
q	= dynamic pressure
$s(x)$	= local semispan at given chordwise station
V	= freestream velocity
u,v,w	= perturbation velocity components
$\bar{w}$	= spanwise average of the vertical velocity component
X_{cp}	= chordwise location of the center of pressure
x,y,z	= Cartesian coordinates
α	= angle of attack
β	$=(\lvert 1-M^2 \rvert)^{1/2}$
ϕ	= perturbation velocity potential
γ,δ	= exponents in the chordwise correction function [Eq. (12)]
Λ	= leading-edge sweep angle, $m=\tan(\pi/2-\Lambda)$
ξ,η	= integration variables
σ	= three-dimensional compressibility correction to the two-dimensional potential [Eqs. (1) and (3)]

Introduction

THE purpose of Ref. 1 (to be published in a forthcoming paper) was to develop an approximate, simple, and inexpensive method for evaluating wing-body interactions in subsonic and supersonic flows. It was to be suitable for slender configurations, such as a typical missile and certain high-speed aircraft configurations. Existing evaluation methods suffer from several shortcomings. Relatively accurate methods, such as Nielsen's[2] or Ferrari's,[3] are limited to a small number of configurations by specifically simplified boundary conditions. Wide-scope numerical methods, such as the Woodward codes,[4] are too costly for use in the preliminary design and parametric investigation stages of a new configuration. The simplest method by far is the slender-body theory,[5] but it is, of course, limited to slender bodies and neglects compressibility effects. It was therefore decided to extend the slender-body method to include chordwise and compressibility corrections so that it could be applied to a wider range of configurations and flight Mach numbers. A relatively simple way to achieve this goal was to adopt a similar Lomax and Sluder correction to slender-wing theory[6] (to make the theory applicable to delta and rectangular wings) and to extend it to slender-body theory.

The three-dimensional wing-body configuration was transformed into a flat wing that could be dealt with, in principle, by the Lomax and Sluder method. However, this led to a problem because the transformation from the three-dimensional to the two-dimensional space resulted in a wing of an arbitrary planform, and the Lomax-Sluder method was developed for either rectangular or triangular wings with straight leading edges. Therefore, the work reported here was necessary before the Lomax-Sluder chordwise and compressibility corrections could be extended to wings of any arbitrary planform.

Analysis

General Description

In the integral equation for the perturbation velocity potential in the thin-wing theory, as derived in Ref. 6 following the general solution of Laplace's equation in Lamb's derivation,[7] the chordwise and compressibility effects are concentrated in a single term. That term can be regarded as a correction term for the two-dimensional doublet distribution, formulated by the slender-wing theory that is independent of chordwise and Mach number effects. Differentiation of the potential equation with respect to (z) results in an integral equation for the vertical perturbation velocity component (w). Using the tangency boundary condition on the wing, this equation is solved for the unknown doublet distribution. Lomax and Sluder used the two-dimensional spanwise pressure distribution, derived by the slender-wing theory for every chordwise station, modified by an unknown longitudinal correction function. By spanwise averaging of the boundary value for the vertical velocity component on the wing, they obtained an integral equation for this correction function.

A similar method is developed here for arbitrary planform shapes. The character of the unknown correction function is determined from an analysis of the singular integrals that

Received June 22, 1981; revision received Nov. 17, 1981.

*NRC Research Associate. Member AIAA.
†Associate Professor, Department of Aeronautical Engineering. Member AIAA.

occur in the derivation of this method and from physical considerations. The coefficients of the function are then determined numerically.

Subsonic Flow

The following analysis is similar to that of Ref. 6 except for the boundaries of the spanwise integrals.

The integral equation for the potential of the perturbation velocity in subsonic thin-wing theory is

$$\phi(x,y,z) = \frac{zV}{8\pi}\int_0^c \left(\int_{-s(x)}^{s(x)} \frac{\Delta p/q}{(y-\eta)^2+z^2} \times\left\{1+\frac{x-\xi}{[(x-\xi)^2+\beta^2(y-\eta)^2+\beta^2z^2]^{1/2}}\right\}\mathrm{d}\eta\right)\mathrm{d}\xi \tag{1}$$

In the slender-wing theory it is

$$\phi(x,y,z) = \frac{zV}{8\pi}\int_{-s(x)}^{s(x)} \frac{\Delta p/q}{(y-\eta)^2+z^2}\mathrm{d}\eta \tag{2}$$

where $\Delta p/q$ is the two-dimensional pressure coefficient. Comparing the two expressions, the term

$$\sigma = 1+\frac{x-\xi}{[(x-\xi)^2+\beta^2(y-\eta)^2+\beta^2z^2]^{1/2}} \tag{3}$$

can be regarded as a three-dimensional, compressibility correction factor for the two-dimensional doublet distribution of the slender-wing theory [Eq. (2)]. The same logic applies also to the pressure coefficient, where the two-dimensional expression can be modified by a chordwise correction function. Following Ref. 8, the pressure coefficient is given by

$$\frac{\Delta p}{q} = -\frac{4}{V}\frac{ws(\xi)}{[s^2(\xi)-\eta^2]^{1/2}}\frac{\mathrm{d}s(\xi)}{\mathrm{d}\xi}f(\xi) \tag{4}$$

where $f(x)$ [or $f(\xi)$] is the chordwise correction function [$f(x)=1$ for the slender-wing theory], and $s(x)$ [or $s(\xi)$] is the local wing semispan.

Spanwise averaging of the vertical velocity component in the wing plane, using the operator

$$\lim_{z\to 0}\frac{1}{2s(x)}\int_{-s(x)}^{s(x)}\frac{\partial\phi}{\partial z}\mathrm{d}y \tag{5}$$

leads to an integral equation for $\bar{w}$

$$\bar{w} = -\frac{V}{8\pi}\int_0^c\int_{-s(\xi)}^{s(\xi)}\frac{\Delta p}{q}\frac{\mathrm{d}\eta}{s^2(\xi)-\eta^2}\mathrm{d}\xi - \frac{1}{2s(x)}\frac{V}{8\pi}\int_0^c\int_{-s(\xi)}^{s(\xi)}\frac{\Delta p}{q}\frac{1}{x-\xi} \times\left\{\frac{[(x-\xi)^2+\beta^2(s(x)-\eta)^2]^{1/2}}{s(x)-\eta} + \frac{[(x-\xi)^2+\beta^2(s(x)+\eta)^2]^{1/2}}{s(x)+\eta}\right\}\mathrm{d}\eta\mathrm{d}\xi \tag{6}$$

Substituting Eq. (4) for $\Delta p/q$ in Eq. (6) and using the notation $s=s(x)$ and $s'=s(\xi)$ one gets

$$\bar{w} = \frac{w}{2\pi}\int_0^x\frac{\pi s'}{s(s^2-s'^2)^{1/2}}\frac{\mathrm{d}s'}{\mathrm{d}\xi}f(\xi)\mathrm{d}\xi + \frac{1}{2s}\int_0^c\frac{s'(\mathrm{d}s'/\mathrm{d}\xi)f(\xi)I_1(x,\xi)}{x-\xi}\mathrm{d}\xi \tag{7}$$

where

$$I_1(x,\xi) = \frac{1}{2}\int_{-s'}^{s'}\frac{1}{(s'^2-\eta^2)^{1/2}}\left\{\frac{[(x-\xi)^2+\beta^2(s-\eta)^2]^{1/2}}{s-\eta} + \frac{[(x-\xi)^2+\beta^2(s+\eta)^2]^{1/2}}{s+\eta}\right\}\mathrm{d}\eta \tag{8}$$

Supersonic Flow

The perturbation velocity potential equation for a thin wing in a supersonic flow can be satisfied by a dipole distribution. Following Ref. 9, the vertical velocity component on the wing surface is

$$w = -\frac{\beta V}{4}\frac{\Delta p}{q} + \frac{V}{4\pi}\mathrm{Re} \times\int_0^x\int_{-s'}^{s'}\frac{(x-\xi)(\Delta p/q)\mathrm{d}\eta\mathrm{d}\xi}{(y-\eta)^2[(x-\xi)^2-\beta^2(y-\eta)^2]^{1/2}} \tag{9}$$

Re stands for the real part of the integral only, thus taking into account only the part of the wing that is within the upstream Mach cone of point (x,y), and the integration must be carried out as indicated. Again, substituting Eq. (4) for the pressure coefficient and applying the averaging operator [Eq. (5)] one gets

$$\bar{w} = \frac{\beta\pi w f(x)(\mathrm{d}s/\mathrm{d}x)}{2} + \frac{w}{2\pi s}\int_0^x\frac{s'(\mathrm{d}s'/\mathrm{d}\xi)f(\xi)I_2(x,\xi)}{x-\xi}\mathrm{d}\xi \tag{10}$$

where

$$I_2(x,\xi) = \mathrm{Re}\int_{-s'}^{s'}\left\{\frac{[(x-\xi)^2-\beta^2(s-\eta)^2]^{1/2}}{s-\eta} + \frac{[(x-\xi)^2-\beta^2(s+\eta)^2]^{1/2}}{s+\eta}\right\}\frac{\mathrm{d}\eta}{(s'^2-\eta^2)^{1/2}} \tag{11}$$

Chordwise Correction Function

The solution of the whole problem hinges now on a solution for the chordwise correction function $f(x)$. Lomax and Sluder[6] proposed such a solution for triangular wings, based on an analysis of the character of the integral equation for $\bar{w}$ and on the requirements that the correction function had to satisfy. Although the equations given here for $\bar{w}$ for the general wing are somewhat different from the equations of Ref. 6 for triangular wings, their character has not changed; therefore, the same character of the solution for $f(x)$ applies here as well.

In subsonic flow the wing boundary conditions require[6] a solution of the form

$$f(x') = \frac{1}{(x')^\delta}(1-x')^{\gamma/2(\gamma+1-x')}(a_0+a_1x'+a_2x'^2+\ldots) \tag{12}$$

where x' is the dimensionless length $x'=x/c$. It is shown in Ref. 6 that this is the correct form for the function, if it is to satisfy the integral equation that has singularities of both the

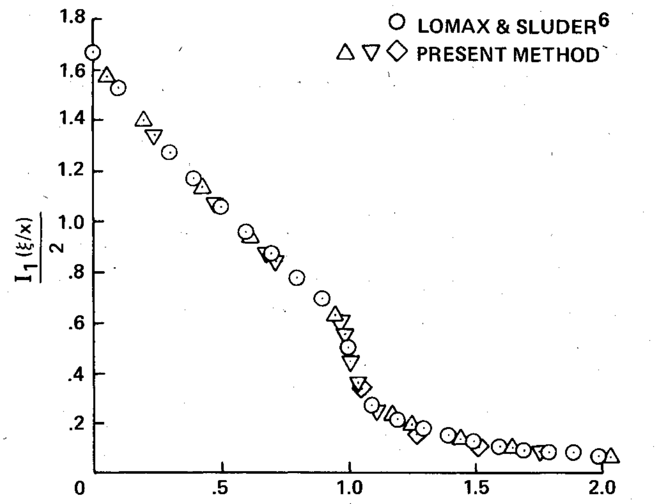

Fig. 1 Comparison of the analytical and numerical solutions of Eq. (8) for a delta wing.

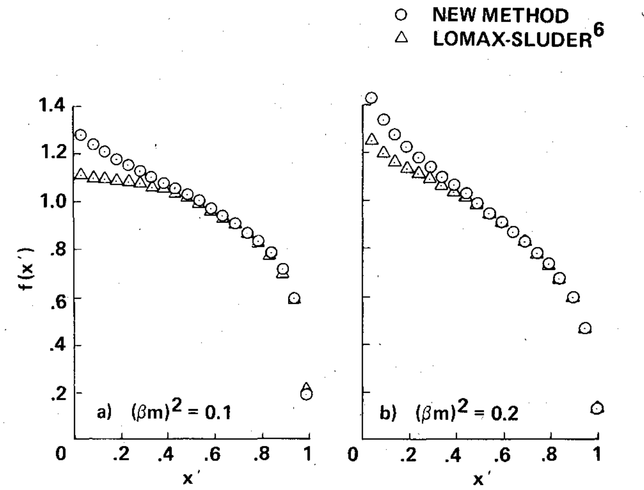

Fig. 2 Comparison of the values of the chordwise correction function $f(x')$ in subsonic flow from Eq. (12) for delta wings with those from Ref. 6: a) $(\beta m)^2 = 0.1$; b) $(\beta m)^2 = 0.2$.

Abel and Cauchy types. This function also has the typical wing leading-edge ($x' = 0$) singularity and satisfies the Kutta condition at the trailing edge [$f(1) = 0$]. These characteristics are equally adequate for both the general wing and the Lomax-Sluder case when $s(x)$ is replaced by (mx) where m is the slope of the leading edge. It is clear, however, that the parameters δ and γ, and the coefficients a_0, a_1, a_2,..., in Eq. (12) will have to be of a higher order in order to better fit the required values at the control points.[6]

In supersonic flow the requirements from the correction function $f(x')$ are much simpler because there is no Kutta condition. These requirements are satisfied by the series

$$f(x') = a_0 + a_1 x' + a_2 x'^2 + ... \quad (13)$$

Solution

The exact and tedious details of the numerical integration of the integrals I_1 and I_2 [Eqs. (8) and (11)] and of the integral equations [Eqs. (7) and (10)] as well as the numerical evaluation of the correction functions [Eq. (12) and (13)] are presented in Ref. 1. The following is but a brief and general description of the process.

Solving Eqs. (7) and (8) in the subsonic flow presents two additional difficulties, compared with the Lomax-Sluder solution of Ref. 6. The equivalent of Eq. (8) in Ref. 6 can be integrated analytically because of the linear relation between the boundaries of the integral ($\pm s'$) and x for a triangular wing. For an arbitrary planform such a linear relation does not exist, and the integral has to be evaluated numerically. The difficulties arise because of the three singularities in the integral at $\eta = \pm s$ and $\eta = s'$; straightforward numerical integration is impossible. The difficulties are overcome in the following way. The integrand is calculated at several points (the more the better) along the integration path, and is then replaced by a cubic spline fit throughout the integration interval.[10] Thus the singularities are avoided with no significant effect on the area below the curve. Special care is needed in fitting the spline function near the singular points, in order to prevent the occurrence of oscillations when the function is forced to change from large-positive to large-negative values or vice versa. This method of integration was checked out on a straight triangular wing by a comparison with the analytical integration of Ref. 6. The results of the comparison are presented in Fig. 1; the agreement is very good.

Once the internal integral I_1 [Eq. (8)] is calculated, the same spline fitting technique can be applied to the chordwise integration [Eq. (7)] to overcome its singularities. This equation includes the unknown correction function $f(\xi')$ (where $\xi' = \xi/c$). As previously stated, the Lomax-Sluder form for f is chosen here also [Eq. (12)]. The linear relation between s and x in the triangular wings of Ref. 6 enabled there a separate determination of the exponent (δ). For an arbitrary planform this is impossible. However, the solution of Ref. 6 for δ shows that δ is related to (βm) (or $\beta Æ$, which is the compressible scaling of the aspect ratio). It is assumed here that for an arbitrary planform (δ) can be determined from (βm), as was done in Ref. 6. This is equivalent to the assumption that the flow in the vicinity of the leading edge of a nontriangular wing resembles the flow near the leading edge of a tangent-triangular wing at the same point. Subsequent calculations with variations of this value of (δ) showed no significant effect on the overall solution, so that this approximation was adopted without further investigation.

With δ fixed, the exponent γ and the coefficients (a_i) have to be determined. Since the formulation is nonlinear for γ and linear for the a_i, a guess is used for the value of γ and a set of n linear equations is obtained for (a_0) to (a_{n-1}) by the calculation of the integral equation at n chordwise points. Naturally, a different set of coefficients is obtained for every guess of γ. Each solution is then matched with the correct value at an additional control point. Many chordwise points were investigated for this purpose, and experience showed that with a second-order polynomial the best computation points are $x' = 0.1$, 0.5, 0.95; the most sensitive control point is at $x' = 0.8$. The value of γ and the corresponding (a_i) set that generate the smallest error at the control point are chosen for further use of the correction function in the evaluation of the lift coefficient and center of pressure.

The accuracy of the computational method was tested by comparing the values of the chordwise correction function $f(x')$ as calculated here with the values from Ref. 6 for delta wings in subsonic flow. The comparison is presented in Fig. 2. There are differences between the two solutions in the upstream 40% of the chord. However, the chordwise matching of the solution of the present method is better than that of Ref. 6. The largest mismatch here at the most sensitive control point is only 0.16% for $\beta m = 0.1$ and 0.02% for $\beta m = 0.2$; it is 2% in Ref. 6. Because the computational accuracy is higher in the present work, it is assumed that the overall results are also better. The same comparison is also carried out for a delta wing in supersonic flow. The numerical integration of Eqs. (10) and (11) is much easier than in the subsonic case. The only singularity in Eq. (11) is of 0(1/2) at the boundary of the integration interval, but the integrand vanishes there. This has a clear significance: the integral has a meaning of a weighted influence on point x of a strip of width dy', but when the flow is supersonic such a strip at $\xi=x$ has no influence on x. The integration of Eq. (10) is also simpler once I_2 is known. The singularity at $x=\xi$ is weak because the numerator vanishes there $[I_2(\xi=x)=0]$. Without the Kutta condition, the chordwise correction function becomes a simple polynomial [Eq. (13)], and its coefficients are determined from the calculation of the integral at three points along the chord.

The correction function for a supersonic delta wing ($m=0.4$, $\beta=0.2$) is compared with the results of Refs. 6 and 11 in Fig. 3. Lomax and Sluder[6] have shown that for a triangular wing in supersonic flow $f(x)=$ const. In this case their value is $f\equiv 0.708$. Stewart[11] has reached the same conclusion by different means and has shown that $f=1/E(1-\beta^2m^2)^{1/2}$ where $E(\ )$ is a complete elliptic integral of the second type. Stewart's value for this case is $f\equiv 0.70512$. The function computed here varies from 0.71 at $x'=0$ to $f=0.70741$ at $x'=1$. The averaged value over the whole wing is

$$\int_0^1 f(\xi')\mathrm{d}\xi' = 0.70882$$

which is fairly close to the Lomax-Sluder result. The apparent reason for this discrepancy is the selection of $x'=0.1$ as one of the points for the determination of the coefficients of $f(x)$ in Eq. (13). The fitting of the function is good (within less than 0.2%) only for $x'>0.1$; however, this point, $x'=0.1$, is important for the nontriangular wings that are computed later, and it was therefore used here also.

Nontriangular Wings

The computational method proposed here is applied next to the family of Gothic and arrowhead wings shown in Fig. 4. The lift coefficient slope C_{L_α} at Mach numbers 0.0, 0.6, 0.8 for these wings is shown in Fig. 5. Also shown for comparison is the variation of C_{L_α} with Mach number according to Jones-Cohen formula.[12]

$$C_{L_\alpha} = \frac{2\pi \mathcal{R}}{p\mathcal{R}+2} \tag{14}$$

where $p=$ wing semiperimeter/wingspan. In order to use Eq. (14) for subsonic compressible flow, the Prandtl-Glauert similarity rule has to be applied. For a triangular wing, Eq. (14) becomes

$$\beta C_{L_\alpha} = \frac{\pi\beta\mathcal{R}}{1+\beta\mathcal{R}/4+[1+(\beta\mathcal{R}/4)^2]^{1/2}} \tag{15}$$

For wings with an arbitrary planform, p has to be calculated separately according to the axis transformation involved in applying the similarity rule.

Included in Fig. 5 are the results for the same wings obtained by the more elaborate and costly vortex-lattice method (VLM) and the Kernel function method (KFM). The computer codes used for this comparison are similar to those of Refs. 13 and 14, respectively. The results obtained by slender-wing theory are also shown.

The slender-wing theory values are much higher than those obtained with the other computations. For $\mathcal{R}=1.08$ the difference is small, but for the higher $\mathcal{R}$ wings, the only region where the slender-wing theory values are near the others is where $M\to 1$; there the computations show a trend of approaching the slender-wing theory results. There are only small differences between the results of Eq. (14), the VLM, and the present method. Equation (14) gives values that are slightly higher than the present method, which corresponds well with the results of Ref. 6. The spread of the results increases as the $\mathcal{R}$ increases. The results of the present method, the VLM, and those of Eq. (14) have all the same character; those of the KFM cut across all the others.

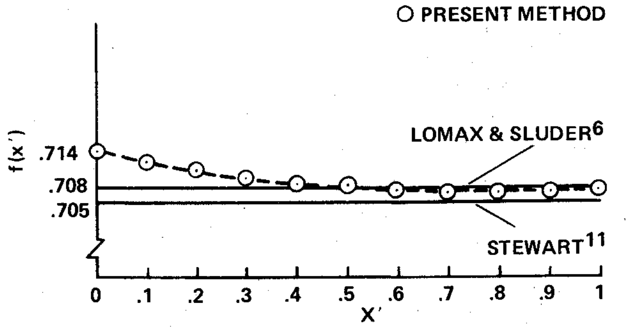

Fig. 3 Comparison of the values of the correction function $f(x')$ from Eq. (13) for a delta wing in supersonic flow with those of Refs. 6 and 11.

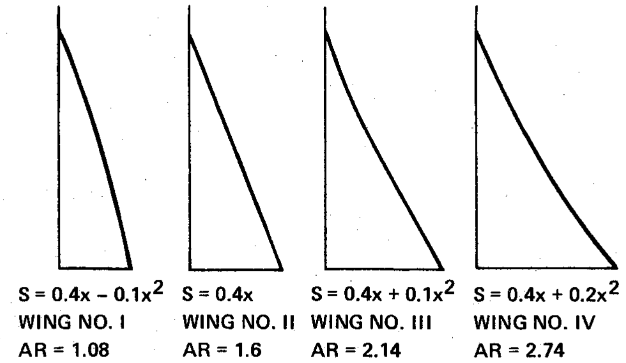

Fig. 4 Family of Gothic and arrow-shaped planforms.

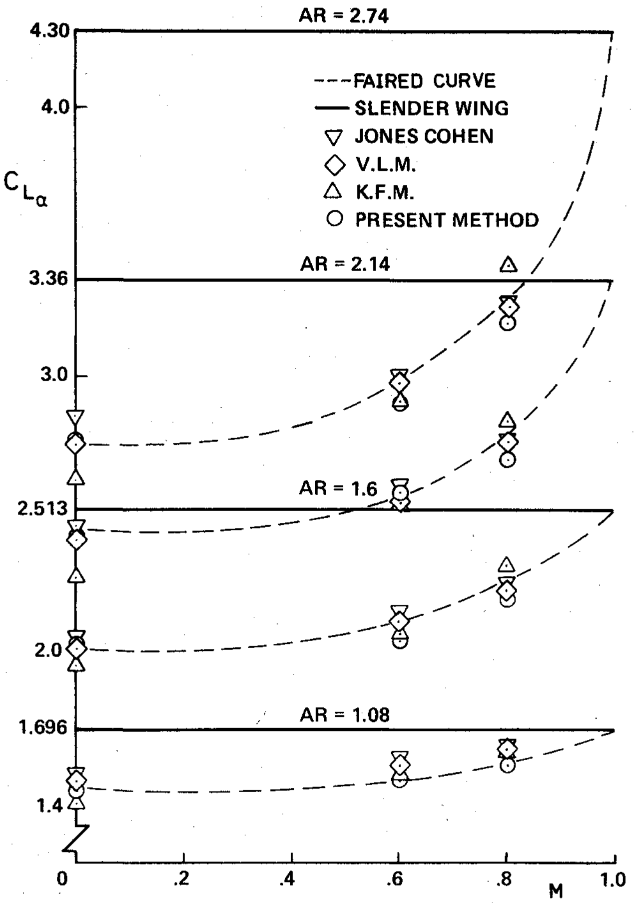

Fig. 5 Comparison of the lift coefficient curve slope results for several Gothic and arrow-shaped wings with other methods.

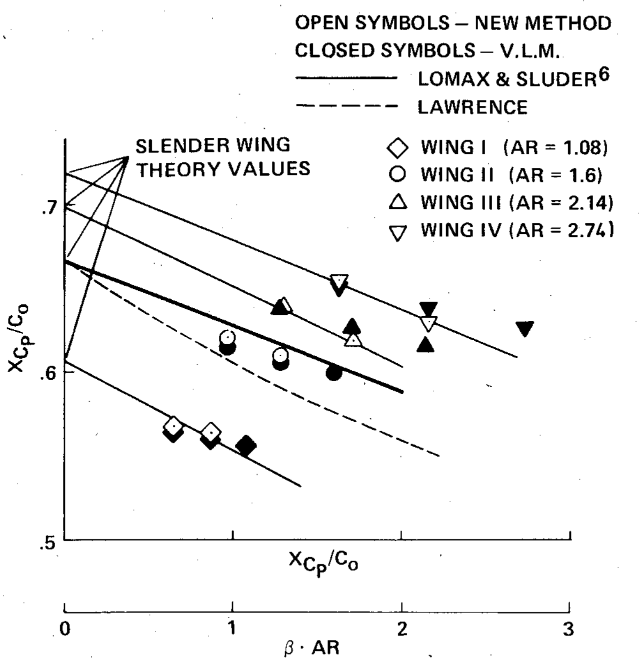

Fig. 6 Variation of the center of pressure X_{cp}/c_0 with the compressibility-scaled aspect ratio $\beta Æ$ in subsonic flow.

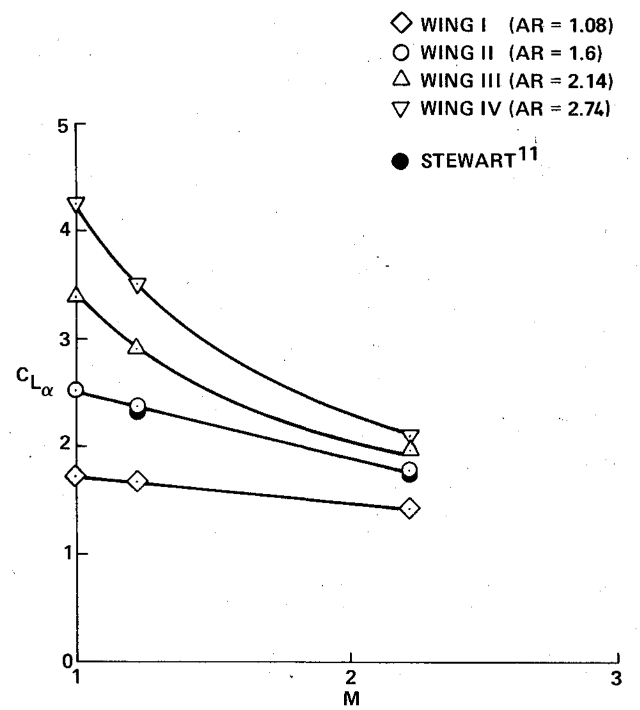

Fig. 7 Lift-coefficient curve slope (C_{L_α}) variation with Mach number for various planforms at supersonic flow.

The variation of the center of pressure (X_{cp}/c_0), where c_0 is the root chord of the wing, with the compressibility-scaled $\beta Æ$ in subsonic flow is presented in Fig. 6 and compared with the VLM results, the results of Ref. 6, and slender-wing theory. Again, there is a good agreement between the proposed method and the VLM results. None of the aforementioned methods could provide results for the supersonic cases. Since there are no other data available for the Gothic and arrowhead wings, the comparison is made for the delta wing only; as in Ref. 6 the agreement with Stewart's[11] results is good (Fig. 7).

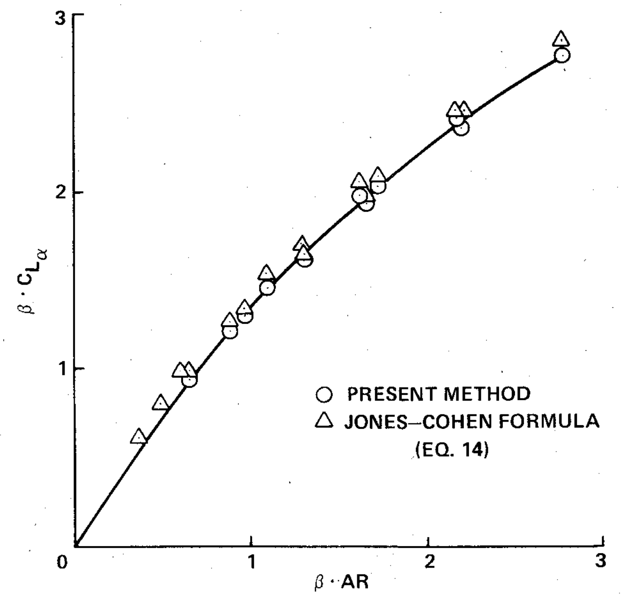

Fig. 8 Compressibility-scaled lift-coefficient curve slope βC_{L_α} variation with reduced aspect ratio $\beta Æ$.

The most significant results of this work are presented in Fig. 8. A single curve correlates the lift-coefficient curve slope for all the different wing planforms, aspect ratios, and Mach numbers, in the subsonic flow region. When plotted as the Mach-number-scaled lift coefficient (βC_{L_α}) vs the Mach-number-scaled aspect ratio ($\beta Æ$) all the data collapse on a single curve. This enables the correlation that was proposed by Lomax and Sluder[6] for delta wings to include wings with arbitrary planform shapes. The results from the Jones-Cohen formula have similar, although slightly higher values.

Conclusion

The Lomax-Sluder chordwise and compressibility correction to slender wing theory is successfully extended to include wings of nontriangular shapes. Although the aspect ratios involved are small, they are too high for the simple slender-wing approximation. The method results in a universal curve for the prediction of the lift-coefficient curve slope for an arbitrary planform wing in subsonic flow (at small angles of attack). It also furnishes a reasonably good estimate of the location of the center of pressure of the wing.

Acknowledgment

The work described herein was done in partial fulfillment of the requirements for the doctor of science degree at Technion—Israel Institute of Technology of the first author.

References

[1] Levin, D., "An Improved Solution for Body-Wing Interaction at Subsonic and Supersonic Speeds," D.Sc. Thesis, Dept. of Aeronautical Engineering, Technion, Israel Institute of Technology, Haifa, Israel, April 1978.

[2] Nielsen, J. N., "Quasi-Cylindrical Theory of Wing-Body Interference at Supersonic Speeds and Comparison with Experiments," NACA 1252, 1955.

[3] Ferrari, C., "Interference between Wing and Body at Supersonic Speeds, Theory and Numerical Application," *Journal of the Aerospace Sciences,* Vol. 15, June 1948, p. 317.

[4] Woodward, F. A., "An Improved Method for the Aerodynamic Analysis of Wing-Body-Tail Configurations in Subsonic and Supersonic Flow," NASA CR-2228, Vols. I and II, May 1973.

[5] Spreiter, J. R., "The Aerodynamic Forces on Slender Plane and Cruciform-Wing and Body Combinations," NACA Rept. 962, 1950.

[6] Lomax, H. and Sluder, L., "Chordwise and Compressibility Corrections to Slender Wing Theory," NACA 1105, 1952.

[7]Lamb, H., *Hydrodynamics,* 6th Ed., Cambridge University Press, 1913, p. 60.
[8]Heaslet, M. A., Lomax, H., and Spreiter, J. R., "Linearized Compressible Flow Theory for Sonic Flight Speeds," NACA TN-1824, 1949.
[9]Lomax, H., Heaslet, M. A., and Fuller, B., "Integrals and Integral Equations in Linearized Wing Theory," NACA 1054, 1951.
[10]Walsh, J. L., Ahlberg, J. H., and Nilson, E. N., "Best Approximation Properties of the Spline Fit," *Journal of Mathematics and Mechanics,* Vol. 11, No. 2, 1962, p. 225.
[11]Stewart, H. J., "The Lift of a Delta Wing at Supersonic Speeds," *Quarterly of Applied Mathematics,* Vol. 4, Oct. 1946, pp. 246-254.
[12]Jones, R. T. and Cohen, D., *High Speed Wing Theory,* Princeton University Press, 1962, p. 96.
[13]Rom, J., Almosnino, D., and Zorea, C., "Calculation of the Non Linear Aerodynamic Coefficients of Wings of Various Shapes and Their Wakes, Including Canard Configurations," Vol. I, *Proceedings of the XI Congress of the International Council of Aeronautical Sciences (ICAS),* Lisbon, Portugal, Sept. 1978, pp. 333-364.
[14]Nissim, E. and Lottati, I., "Oscillatory Subsonic Piecewise Continuous Kernel Function Method," *Journal of Aircraft,* Vol. 14, June 1977, p. 515.